Comparative Study of Restoration Algorithms ISTA and IISTA

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Abstract—our proposed work is to compare iterative shrinkage thresholding algorithm (ISTA) and improved iterative shrinkage thresholding algorithm (IISTA) for image restoration of linear inverse problems. This class of problems results from combining a linear observation model with a non-quadratic regularizer e.g., total variation regularization. Number of iterations, Regularization parameters and Regularization functions are used for comparsion.Thsese algorithms are performed through a recursive application of two simple procedures linear filtering and soft thresholding.Experimental results shows better performance of IISTA than ISTA.

Index Terms— IISTA,ISTA, image restoration, inverse problems, l_0 norm,l_1 norm, l_2 data fidelity term, regularization function ,total variation

1 INTRODUCTION

INVERSE problems abound in many application areas of image processing, remote sensing, radar imaging,tomographic imaging, microscopic imaging, astronomic imaging ,digital photography [1][2][3][16]. Image restoration is one of the earliest and most classical linear inverse problems in imaging dating back to the 1960s [1].In a inverse problem, the goal is to estimate an unknown original image **x** from a noisy observation **y,** produced by an operator **K** applied to **x**, when **K** is a linear we have linear inverse problems (LIP). Many applications to LIPs define a solution x a restored image as a minimizer of a convex objective function $f: x \to \mathbb{R} = [-\infty, +\infty]$ given by

$$f(x) = 1/2 \parallel y - Kx \parallel^2 + \lambda\phi(x) \qquad (1)$$

where K: X→Y is the linear direct operator, X and Y are real Hilbert spaces (both with norm denoted as $\|\cdot\|$), ϕ :X→R is a function, λ∈ [0, +∞] is a parameter.

In a regularization framework minimizing f is seen as a way of overcoming the ill-conditioned or singular nature of K, which precludes inverting it. In this context ϕ is called regularizer and λ is called regularization parameter [5].
In a finite dimensional Bayesian setting, the reasoning behind (1) as follows: Assume that y=kx+w, where w is a sample of a white zero-mean Gaussian random vector/field of variance σ^2 , let p(x) be the adopted prior, thus the logarithm of a posteriori density is $\log p(x|y) = -f(x)$ upto a constant with $\lambda=\sigma^2$ and $\phi(x) = -\log p(x)$: maximum posteriori (MAP) estimate are thus minimizer of f. Despite the possible interpretation of (1)
We will refer to ϕ simply as the regularizer.The intuitive meaning of f is simple: minimizing it corresponds to looking for a compromise between the lack of fitness of a candidate estimate X to the observed data, measured by $\parallel y - Kx \parallel^2$ and its degree of undesirability given by $\phi(x)$. The regularization parameter λ controls the relative weight of the two terms. Examples of total variation (TV) regularization [8][17] and wavelet based regularization [18][19].The non-differentiable nature of f together with the huge dimension of its argument for example 512x512 image X=R[262144], place its minimization beyond the reach of standard off-the-shelf optimization methods.
This paper strictly concerned with algorithms for minimization [1] and discusses different choices of λ and ∅.

2 PRELIMINARIES

2.1 Total Variation Function

For a real valued continuous function f defined in an interval $[a, b] \subset R$ its total variation is one measure of a one dimensional arc length of the curve with parametric equation $x \to f(x)$ for $x \in [a, b]$ [8][9][10][11].
The total variation of differential function f defined on an interval $[a, b] \subset R$ has the following expression f' is Riemann integral.

$$V_b^a(x) = \int_a^b |f'(x)| \; dx \qquad (2)$$

The form of the total variation of differentiable function of several variables. Given a differential function f defined on a bounded open set $\Omega \subseteq R^n$ the total variation of f has the following expression

$$V(f, \Omega) = \int_\Omega |\nabla f(x)| \; dx \qquad (3)$$

Here $|\cdot|$ denotes the l_2 norm
Proof: The first step in the proof is to prove as equality which follows from Gauss-Ostrogradsky theorem

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Lemma:
Under the conditions of the theorem the following equality holds

$$\int_\Omega f\, div\varphi = -\int_\Omega \nabla f \cdot \varphi \qquad (4)$$

Proof of the Lemma:

From Gauss-Ostrogradsky theorem

$$\int_\Omega div\, R = \int_{\partial\Omega} R\cdot n \qquad (5)$$

By substituting R=fφ we have

$$\int_\Omega div(f\varphi) = \int_{\partial\Omega} f\varphi \cdot n \qquad (6)$$

Where φ is zero on the border of Ω by definition

$$\int_\Omega div(f\varphi) = 0 \qquad (7)$$

$$\int_\Omega \partial x_i\,(f\varphi_i) = 0 \qquad (8)$$

$$\int_\Omega (\varphi_i \partial x_i f + f\ \partial x_i \varphi_i) = 0 \qquad (9)$$

$$\int_\Omega f\ \partial x_i \varphi_i = -\int_\Omega \varphi_i\, \partial x_i\ f \qquad (10)$$

$$\int_\Omega f\, div\,\varphi = -\int_\Omega \varphi \cdot \nabla f \qquad (11)$$

Proof the equality under the condition of the theorem, from the lemma we have

$$\int_\Omega f\, div\,\varphi = -\int_\Omega \varphi \cdot \nabla f \le \left|\int_\Omega \varphi \cdot \nabla f\right|$$

$$\le \int_\Omega |\varphi| \cdot |\nabla f| \le \int_\Omega |\nabla f| \qquad (12)$$

in the last part φ could be omitted because by definition its essential supermum is at most one

On the other hand we consider θ_n: $II_{[-N,N]}\frac{\nabla f}{|\nabla f|}$ and θ_n^* which is up to ε approximation of $\theta in C_c'$ with the same integral. We can do this since C_c' is dense inL'. Now again substituting into the lemma

$$\lim_{N\to\infty}\int_\Omega f div\theta_n^* = \int_\Omega II_{[-N,N]}\nabla f \cdot \frac{\nabla f}{|\nabla f|} = \lim_{N\to\infty}\nabla f \cdot \frac{\nabla f}{|\nabla f|}$$

$$= \int_\Omega |\nabla f| \qquad (13)$$

This means we have a convergent sequence of $\int_\Omega f div\varphi$ that tends to $\int_\Omega |\nabla f|$ as well as know that

$$\int_\Omega f\, div\,\varphi \le \int_\Omega |\nabla f| \qquad (14)$$

It can be seen that from the proof that the supermum is attained when $\varphi \to -\frac{\nabla f}{|\nabla f|}$The function f is said to be of bounded variation precisely if its total variation is finite.

2.2 The General Optimization Model

Given min {F(**x**) =f(**x**) +g(**x**): **x** ϵ E} (15)
with the following assumptions.

1. The vector space E stands for a finite dimensional Euclidean space with inner product$<\cdot,\cdot>$ and norm $\|\cdot\| = <\cdot,\cdot>^{1/2}$.

2. g :E→ $(-\infty, +\infty]$ is a proper closed convex function.

3 .f: E→R is a continuously differentiable with Lipschitz continuous gradient L (f)

$\|\nabla f(x) - \nabla f(y)\| = L(f)\,\|x - y\|$ for every x, y ϵ E where$\|\cdot\|$ denotes standard Euclidean norm andL L (f) >0 is the Lipschitz constant [10] [11] of ∇f.

4 Equation (15) is solvable X_*=argmin F≠ 0 and for $\mathbf{x}^* \epsilon$ X, we set $F_* = F(x^*)$.
In particular, the standard convex constrained minimization problem:
Min {f(x): **x** ϵ C} (16)
is obtained by choosing g(**x**) =δ_c(**x**), with C ⊆E some closed convex set and $\delta_c$being the indicator function on C. Likewise, with f(**x**)≡0, the general non smooth convex minimization problem is obtained.[13]

2.3 Proximal Map

The key role within approach is the proximal map of Moreau associated to a convex function.
Given proper closed convex function: E→ $(-\infty, +\infty]$ and any scalar t>0, the proximal map associated to g is defined by

$$\mathbf{prox_t(g)(x)} = arg^{min}_x\{g(u) + 1/2t\ \|u - x\|^2\} \qquad (17)$$

The proximal map associated with a closed proper convex func-

tion g has the following properties.
Lemma: Let g→ $(-\infty, +\infty]$ be a closed proper convex function and for any t>0,

$$g_t(x) = \inf_x \{g(u) + 1/2t \,\| \mathbf{u} - \mathbf{x} \|^2\} \tag{18}$$

Then, a) The infirmum in (18) is attained at the unique point $\text{prox}_t(g)(x)$. As a consequence, the map $(I + \partial g)^{-1}$ is single valued from E into itself and

$$(I + \partial g)^{-1}(\mathbf{x}) \equiv \text{prox}_t(g)(\mathbf{x}) \forall \mathbf{x} \in E \tag{19}$$

b) The function g_t is continuously differentiable on E with Lipschitz gradient given by

$$\nabla g_t(x) = 1/t\big(I - \text{prox}_t(g)(\mathbf{x})\big) \qquad \forall x \in E \tag{20}$$

c) In particular if g= δ_c , the indicator of a closed convex set C ∈ E, then $\text{prox}_t t(g)\ (\mathbf{x})\ (I + \partial g)^{-1} = \mathbf{P_c}$, the Euclidean projection operator on C and $g_t(\mathbf{x}) = \|P_c(x) - x\|^2$

3 Proposed Work

3.1 Basic Gradient Based Algorithm

The equation (15) solves as follows. Fix any scalar t > 0, then $\mathbf{x}^*$ solves the convex minimization problem if and only if the following statements hold:

$$0 \quad \epsilon \quad t\nabla f(\mathbf{x}^*) + t\partial g(\mathbf{x}^*)$$

$$0 \quad \epsilon \quad t\nabla f(\mathbf{x}^*) - \mathbf{x}^* + \mathbf{x}^* + t\partial g(\mathbf{x}^*)$$

$$(I + \partial g)^{-1}(\mathbf{x}^*) \quad \epsilon \quad (I - t\nabla f)(\mathbf{x}^*)$$

$$\mathbf{x}^* = (I + \partial g)^{-1}(I - t\nabla f)(\mathbf{x}^*) \tag{21}$$

The equation (21) calls for the fixed point iterative scheme:

$$\mathbf{x_0} \,\epsilon\, E, \mathbf{x_k} = (I + \partial g)^{-1}(I - t_k\nabla f)(\mathbf{x_{k-1}}), (t_k > 0) \tag{22}$$

$$\begin{aligned}\mathbf{x_k} &= prox_{t_k}(\mathbf{g})(I - t_k\nabla f)(\mathbf{x_{k-1}}) \\ &= \boldsymbol{arg}\min_{x\in E} \{\mathbf{1/2}\boldsymbol{t_k}\|\boldsymbol{x} - (\boldsymbol{x_{k-1}} - \boldsymbol{t_k}\,\boldsymbol{\nabla f}(\boldsymbol{x_{k-1}}))\|^2 + \boldsymbol{g}(\boldsymbol{x})\end{aligned} \tag{23}$$

3.1 IST Algorithm

when g(x) =$\|\mathbf{x}\|_1$ this equation reduces to

$$\mathbf{x_k} = \boldsymbol{\psi}_{\boldsymbol{\tau\phi}}(\mathbf{x_{k-1}} - t_k\,\nabla f(\mathbf{x_{k-1}})) \tag{24}$$

Equation (24) is called IST (Iterative Shrinkage Thresholding) Algorithm [13], where $\boldsymbol{\psi}_{\boldsymbol{\tau\phi}}: \mathbf{E} \to \mathbf{E}$ is a shrinkage operator defined by $\psi_{\tau\phi} = (|x_i| - \tau)_+ \text{ sign } (x_i)$. A typical condition ensuring the convergence of the sequence x_k produced by IST algorithm is to require $t_k\ \epsilon\ (0, 2/L(f))$.

3.2 Improved IST Algorithm

1. Input an upper bound L ≥ L(f) on the Lipschitz of ∇f
2. Take $y_1=x_0\ \epsilon$ E, $t_1=1$
3. For k≥ 1. Compute $X_k = p_L(y_k)$

$$t_k = (1+\sqrt{1+4t^2})/2$$

$$y_{k+1} = x_k + (t_k - 1)/t_{k+1}\,(X_k - X_{k-1})$$

where $\mathbf{p_L}(\mathbf{y}) = \mathbf{prox_{1/L}}(g)\,(y - 1/L\,\nabla f(y))$

$$= \boldsymbol{arg}\min_{x\in E} \{\boldsymbol{L/2}\,\|\boldsymbol{x} - (\boldsymbol{y} - L/2\,\nabla f(y))\|^2 + \boldsymbol{g}(\boldsymbol{x})\}$$

Our proposed algorithm is called Improved Iterative Shrinkage Thresholding algorithm (IISTA) [14]. In this each iterates depends on the previous two iterate and not only on the last iterate as in ISTA. The operator depends p_L uses two previous iterates (X_k, X_{k-1}) as a linear combination.
The rate of convergence of IISTA is O ($1/k^2$) while it is O (1/k) in ISTA.

3.3 Regularization Functions

In general, the image restoration problems have the form

$$\min_{x\in R^n} \varphi(x) = f(x) + \tau c(x) \tag{25}$$

f:$R^n \to R$ is smooth and convex data fidelity term, usualy

$$f(x) = \frac{1}{2}\,\|Ax - y\|_2^2 \tag{26}$$

C:$R^n \to R$ is a regularization or penalty function, typically convex often non-differentiable.

If A=I, we have a denoising problem.
If c is a proper and convex φ is strictly convex, there is a unique minimizer. Thus the shrinkage thresholding function is

$$\Psi_\lambda = arg\min_z \frac{1}{2}\,\|Z - u\|_2^2 + \lambda c(z) \tag{27}$$

is a well defined Moreau-proximal mapping.

If $c(z) = \|z\|_0$, l_0 norm then $\Psi_\lambda = hard(z,\lambda)$ where $hard(z,\lambda) = x.^*$(abs(x)>=sqrt (2*λ)).

If $c(z) = \|z\|_1$,l_1 norm then $\Psi_\lambda(z) = soft(z,\lambda)$ where

$soft(z,\lambda) = sign.^*\ (|z| - \lambda)_+$and

$$(a)_+ = \begin{cases} 0 & if\ a<0 \\ a & if\ a\geq 0 \end{cases}$$

Note that both functions are component wise application

If c (z) =TV (z), total variation function then equation becomes

$$\Psi_\lambda(z) = arg^{min}_z \frac{1}{2} \parallel Az - y \parallel_2^2 + \tau TV(z) \quad (21)$$

Where $TV(z) = \sqrt{(\Delta_i^h z)^2 + (\Delta_i^v z)^2}$ where Δ_i^h(z) and Δ_i^v(z) are linear operators corresponding to horizontal and vertical first order differences at point i respectively i.e., $\Delta_i^h z = z_i - z_j$ where j is the first order neighbour to the left of i, and $\Delta_i^v z = z_i - z_k$ where k is the first order neighbour above i. This equation is isotropic and not differentiable.

4 Experimental Results

Our Experiments are carried with MATLAB R2010a and laptop of Intel Corei3 processor. The observation shows that objective function monotonically decreasing as the number of iteration increases. The Iterative Signal-to-Noise Ratio (ISNR) increases when the number of iteration increases. The number of iteration required by IISTA for restoration is less than that of ISTA. When the number of iteration increases then the value of objective function decreases

Experiments are performed using blur size 4x4 and 9x9. The increase of blur size shows that increase of iteration for restoration. The IST and IIST algorithms are compared with the role of regularization functions and regularization parameters.

The Number of iterations is compared in the tables 1-4. Tables 5-8 gives result of regularization function role with number of iterations. The study regularization parameter **τ** with ISNR, objective function and regularization function is done in tables 9-12.

Fig 1 and 2 shows the value of ISNR is higher for IISTA than ISTA. Fig 3 and 4 shows restoration using different norms.

5 Conclusion

The performance of IISTA is better than ISTA in terms of number of iterations and rate of convergence. Under blur size 9x9 and 4x4 TV norm shows better ISNR than l_0 norm and l_1 norm for both ISTA and IISTA.In the case of l_1 norm using 9x9 blursize shows recovery is not possible after a certain number of iterations. The study of regularization parameter shows better performance for l_0 norm than TV norm and l_1 norm. It is observed that when the value of **τ** decreases the performance of algorithms is also decreases. Our studies may be continued using different types of noise.

Table 1

Variations of different parameters using ISTA blur size 4x4

Number of Iterations	Objective function(10^4)	Criterion	ISNR	CPU time Seconds
500	1.37235	0.8517	6.794	97.48
1000	1.34262	0.3016	7.155	187.41
1500	1.32274	0.1433	7.446	302.91
2000	1.32123	0.0700	7.652	398.38
2500	1.31348	0.0441	7.722	505.04
3000	1.31317	0.0242	7.793	584.27

Table 2

Variations of different parameters using IISTA blur size 4x4

Number of iterations	objective function(10^4)	Criterion	ISNR	CPU time Seconds
50	1.39000	17.74000	6.649	11.56
100	1.33174	4.20200	7.478	19.59
150	1.31601	0.60700	7.903	30.72
200	1.31479	0.09780	7.968	38.90
250	1.30899	0.08699	7.965	53.16
300	1.301143	0.07359	8.080	57.87

Table 3

Variation of different parameters using ISTA Blur size 9x9

Number of iterations	Objective function(10^3)	Criterion	ISNR	CPU time Seconds
1000	9.24413	0.4265	6.679	184.12
2000	8.94820	0.1473	7.167	364.27
3000	8.89792	0.0762	7.504	537.14
4000	8.87524	0.0459	7.796	726.74
5000	8.79952	0.;032	7.947	932.47

Table 4

Variations of different parameters using IISTA blur size 9x9

Iterations	Objective function (10^3)	Criteri-on	ISNR	CPU time seconds
100	9.16054	5.900	6.974	20.95
150	8.93014	2.846	7.589	28.74
200	8.85125	1.789	8.064	38.35
250	8.890913	0.800	8.415	52.08
300	8.81801	0.331	8.633	59.64
400	8.72853	0.053	8.786	84.01
500	8.77176	0.036	8.846	106.54

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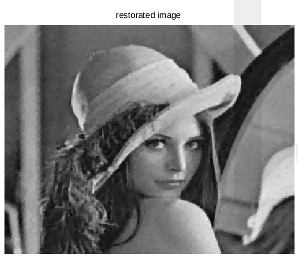
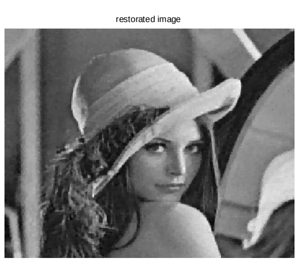
(a)

(b)



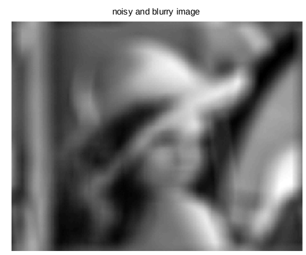
(c)



(d)

Fig. 1 a) original Image b) noisy and blurry Image blur size 4x4 c) Restoration using ISTA ISNR=7.155 number of iterations=1000 d) Restoration using IISTA ISNR=8.004number of iterations=300

(a)



(b)

(c)

(d)

Fig. 2 a) Original Image b) noisy and blurry Image blur size 9x9 c) Restoration using ISTA ISNR=7.947 number of iterations=5000 d) Restoration using IISTA ISNR=8.846 number of iterations=500

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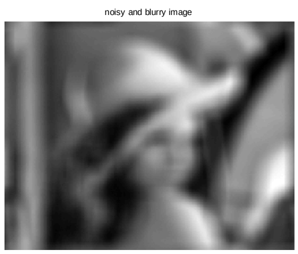
(a)



(b)

(c)

(d)

(e)

Fig 3. a) Original image b) noisy and blur image Blur size 9x9 Restorated using ISTA c) TV norm d) l_1 norm and e) l_0 norm Number of iterations is 5000.

Table 5

ITERATIONS vs. ISNR using ISTA Blur size 9x9

Iterations	ISNR		
	TV norm	l_1 norm	l_0 norm
500	6.275	6.285	6.285
1000	6.690	6.688	6.688
1500	6.985	6.955	6.955
2000	7.210	7.174	7.210
2500	7.398	7.125	7.367
3000	7.557	7.149	7.530
3500	7.655	7.141	7.648
4000	7.786	7.137	7.781
4500	7.865	7.157	7.879
5000	7.993	7.163	7.974

Table 6

ITERATIONS vs. Objective function using ISTA Blur size 9x9

Iterations	Objective function		
	TV norm (10^3)	l_1 norm(10^4)	l_0 norm(10^3)
500	9.54921	2.81700	5.42570
1000	9.23162	2.80714	5.18368
1500	9.09400	2.79831	6.95500
2000	9.05012	2.80361	4.98930
2500	8.91964	2.79718	4.88313
3000	8.92602	2.80016	4.89099
3500	8.92025	2.80091	4.85595
4000	8.92264	2.79913	4.83579
4500	8.80493	2.80014	4.81461
5000	8.84212	2.80633	4.84594

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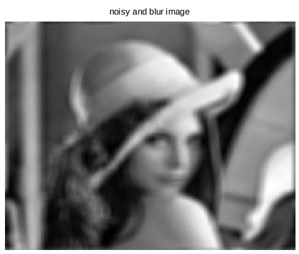
(a)



(b)

(c)

(d)

(e)

Fig 4 a) Original image b) noisy and blur image blur size 4x4 Restorated using IISTA c) TV norm d) l_1 norm and e) l_0 Number of Iterations is 150

Table 7
ITERATIONS Vs ISNR and Objective function using IISTA Blur size 9x9

Iterations	ISNR		Objective function	
	TV norm	l_0 norm	TV norm (10^3)	l_0 norm(10^3)
50	6.613	6.160	9.70883	5.53750
100	6.951	6.972	9.14136	5.02918
150	7.612	7.622	9.01817	4.81195
200	8.077	8.127	8.81832	4.78740
250	8.452	8.410	8.81019	4.79088
300	8.660	8.612	8.81834	4.80032
350	8.452	8.682	8.79188	4.72642
400	8.750	8.801	8.78827	4.76248
450	8.719	8.788	8.80850	4.79617
500	8.736	8.718*	8.79433	4.78165

Table 8
ITERATIONS vs. ISNR using IISTA Blur size4x4

Iterations	ISNR		
	TV norm	l_1 norm	l_0 norm
50	6.649	6.431	6.610
60	6.841	6.523	6.710
70	7.037	6.525	6.753
80	7.130	6.523	6.852
90	7.322	6.544	6.880
100	7.482	6.549	6.903
110	7.560	6.578	6.891
120	7.651	6.554	6.842
130	7.764	6.535	6.891
140	7.823	6.521	6.909
150	7.918	6.606	6.827

Table 9
Regularization parameter **τ** vs. ISNR using ISTA
Number of Iterations=50

τ	ISNR		
	TV norm	l_1 norm	l_0 norm
10^{-1}	6.427	5.916	5.920
10^{-2}	6.002	6.000	6.002
10^{-3}	5.941	5.879	5.930
10^{-4}	5.902	5.923	5.936
10^{-5}	5.894	5.898	5.895
10^{-6}	5.901	5.895	5.932
10^{-7}	5.921	5.894	5.882
10^{-8}	5.914	5.902	5.914
10^{-9}	5.899	5.921	5.916
10^{-10}	5.905	5.914	5.924
10^{-20}	5.850	5.899	5.898
10^{-30}	5.898	5.905	5.891
10^{-38}	5.892	5.899	5.912

Table 10
Regularization parameter **τ** vs. Objective function using ISTA Number of Iterations=50

τ	Objective function		
	TV norm ($x10^4$)	l_1 norm (10^4)	l_0 norm (10^4)
10^{-1}	6.9238	28.1120	1.1537
10^{-2}	1.3130	3.2429	0.5330
10^{-3}	0.5531	0.7370	0.4730
10^{-4}	0.4734	0.4958	0.4620
10^{-5}	0.4570	0.4635	0.4606
10^{-6}	0.4600	0.4560	0.4597
10^{-7}	0.4576	0.4561	0.4589
10^{-8}	0.4600	0.4623	0.4644
10^{-9}	0.4614	0.4576	0.4579
10^{-10}	0.4630	0.4600	0.4629
10^{-20}	0.4584	0.4614	0.4641
10^{-30}	0.4584	0.4629	0.4617
10^{-38}	0.4594	0.4649	0.4556

Table 11
Regularization parameter **τ** vs. ISNR using IISTA
Number of iterations=50

τ	ISNR		
	TV norm	l_1 norm	l_0 norm
10^{-1}	6.780	5.959	5.974
10^{-2}	6.544	6.543	6.554
10^{-3}	5.928	5.913	5.942
10^{-4}	5.890	5.935	5.910
10^{-5}	5.934	5.909	5.898
10^{-6}	5.931	5.917	5.994
10^{-7}	5.906	5.945	5.915
10^{-8}	5.924	5.892	5.917
10^{-9}	5.860	5.926	5.933
10^{-10}	5.907	5.928	5.908
10^{-20}	5.920	5.938	5.910
10^{-30}	5.916	5.890	5.958
10^{-38}	5.934	5.906	5.930

Table 12
Regularization parameter **τ** vs. Objective function using IISTA
Number of iterations=50

τ	Objective function		
	TV norm ($x10^4$)	l_1 norm (10^4)	L_0 norm (10^4)
10^{-1}	5.8518	28.1210	1.1564
10^{-2}	1.2533	3.2316	0. 5159
10^{-3}	0.5021	0.6792	0.4112
10^{-4}	0.4031	0.4319	0.3946
10^{-5}	0.3986	0.3999	0.4002
10^{-6}	0.3994	0.3969	0.3963
10^{-7}	0.3995	0.3956	0.3981
10^{-8}	0.3969	0.3952	0.4023
10^{-9}	0.3952	0.4002	0.3961
10^{-10}	0.3925	0.3938	0.3965
10^{-20}	0.3957	0.3992	0.3959
10^{-30}	0.3991	0.3967	0.3924
10^{-38}	0.3976	0.3995	0.4001

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